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THEORY OF LARGE-STRAIN TORSION OF PRISMATIC BODIES WITH MOMENT STRESSES

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The problem of the torsion and tension–compression of a prismatic bar with a stress-free lateral surface is studied using three-dimensional elasticity theory for materials with moment stresses. A substitution is found that allows one to separate one variable in the nonlinear equilibrium equations for a Cosserat continuum and boundary conditions on the lateral surface. This substitution reduces the original spatial problem of the equilibrium of a micropolar body to a two-dimensional nonlinear boundary-value problem for a plane region shaped like the cross section of the prismatic bar. Variational formulations of the two-dimensional problem for the section are given that differ in the sets of varied functions and the constraints imposed on their boundary values.

Key words: *large strains, moment stresses, nonlinear Saint Venant's problem.*

1. Reduction to Two-Dimensional Boundary-Value Problem. In the absence of mass forces and moments, the system of equations governing the statics of a Cosserat nonlinear-elastic continuum [1, 2] comprises the stress equilibrium equations

$$\operatorname{div} D = 0, \qquad \operatorname{div} G + (C^{\mathrm{t}} \cdot D)_{\times} = 0, \tag{1.1}$$

the constitutive equations

$$D = P \cdot H, \qquad G = K \cdot H,$$
$$P = \frac{\partial W}{\partial Y}, \qquad K = \frac{\partial W}{\partial L}, \qquad W = W(Y, L), \tag{1.2}$$

and the geometrical relations

$$Y = C \cdot H^{\mathrm{t}}, \qquad C = \operatorname{grad} \boldsymbol{R}, \qquad \boldsymbol{R} = X_k \boldsymbol{i}_k, \qquad L \times E = -(\operatorname{grad} H) \cdot H^{\mathrm{t}}. \tag{1.3}$$

Here D and G are the Piola-type stress and moment-stress tensors, respectively, P and K are the Kirchhoff-type stress and moment-stress tensors, respectively, C is the strain gradient, H is the proper orthogonal microrotation tensor characterizing the rotational degrees of freedom of particles of the Cosserat continuum, X_k ($k = 1, 2, 3$) are the Cartesian (Eulerian) coordinates of the deformed body, $\boldsymbol{i}_k$ are the coordinate unit vectors, Y is the strain measure, L is the flexural-strain tensor, E is the unit tensor, and W is the specific free energy of the elastic material; div and grad are the divergence and gradient operators in the Lagrangian coordinates, respectively [below, the Cartesian coordinates of the reference configuration of the body x_s ($s = 1, 2, 3$) are used as the Lagrangian coordinates], and the subscript "×" in (1.1) denotes the vector invariant of the second-rank tensor. Substituting relations (1.2) and (1.3) into (1.1), we obtain a system of six equations with unknown functions X_1, X_2, X_3, and H and independent variables x_1, x_2, and x_3.

In the reference configuration, the elastic body is assumed to have the shape of a cylinder (prism) of arbitrary cross section. The generatrices of the cylinder are parallel to the x_3 axis and the coordinates x_1 and x_2 are reckoned

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in the cross-sectional plane. To reduce the three-dimensional problem of the nonlinear moment theory of elasticity to the two-dimensional problem, we consider the following two-parameter family of strains of the Cosserat continuum:

$$
\begin{aligned}
X_1 &= u_1(x_1, x_2)\cos\psi x_3 - u_2(x_1, x_2)\sin\psi x_3,\\
X_2 &= u_2(x_1, x_2)\cos\psi x_3 + u_1(x_1, x_2)\sin\psi x_3,\\
X_3 &= \lambda x_3 + w(x_1, x_2) \qquad (\lambda, \psi = \text{const});
\end{aligned}
\tag{1.4}
$$

$$
H(x_1, x_2, x_3) = H_0(x_1, x_2)\cdot Q(x_3). \tag{1.5}
$$

Here $H_0^{-1} = H_0^{\mathrm{t}}$, $Q = \boldsymbol{i}_1\otimes\boldsymbol{e}_1 + \boldsymbol{i}_2\otimes\boldsymbol{e}_2 + \boldsymbol{i}_3\otimes\boldsymbol{i}_3$ ($\boldsymbol{e}_1 = \boldsymbol{i}_1\cos\psi x_3 + \boldsymbol{i}_2\sin\psi x_3$ and $\boldsymbol{e}_2 = -\boldsymbol{i}_1\sin\psi x_3 + \boldsymbol{i}_2\cos\psi x_3$ are the unit vectors), and H_0 and Q are the proper orthogonal tensors. The geometrical meaning of representations (1.4) and (1.5) is that the cross section of the prism at a distance x_3 from the coordinate origin is subjected to a certain plane strain defined by the functions u_1 and u_2 and to warping defined by the function w, rotates about the bar axis through a finite angle ψx_3, and moves a distance λx_3 along the axis. Moreover, particles of the body undergo microrotations specified by relation (1.5). Expressions (1.4) and (1.5) extend the representations of the finite torsional deformations proposed in [3] to the case of a medium with moment stresses.

From (1.3)–(1.5), we obtain

$$
C = C_0(x_1, x_2)\cdot Q; \tag{1.6}
$$

$$
Y = C_0\cdot H_0^{\mathrm{t}}, \qquad L = \frac{1}{2}\boldsymbol{i}_\alpha\otimes\Big(\frac{\partial H_0}{\partial x_\alpha}\cdot H_0^{\mathrm{t}}\Big)_\times + \psi\boldsymbol{i}_3\otimes\boldsymbol{i}_3\cdot H_0^{\mathrm{t}}, \tag{1.7}
$$

where

$$
C_0 = \frac{\partial u_\beta}{\partial x_\alpha}\boldsymbol{i}_\alpha\otimes\boldsymbol{i}_\beta + \frac{\partial w}{\partial x_\alpha}\boldsymbol{i}_\alpha\otimes\boldsymbol{i}_3 - \psi u_2\boldsymbol{i}_3\otimes\boldsymbol{i}_1 + \psi u_1\boldsymbol{i}_3\otimes\boldsymbol{i}_2 + \lambda\boldsymbol{i}_3\otimes\boldsymbol{i}_3 \qquad (\alpha, \beta = 1, 2).
$$

Because $Q(0) = E$, the following relations hold:

$$
C_0 = C(x_1, x_2, 0), \qquad H_0 = H(x_1, x_2, 0).
$$

According to (1.7), the strain measure Y and the flexural-strain tensor L do not depend on the coordinate x_3. If the elastic body is homogeneous along the coordinate x_3, it follows from (1.2) that the stress and moment-stress tensors P and K are functions of only the coordinates x_1 and x_2. The homogeneity of the body along the coordinate x_3 implies that the specific free energy W depends explicitly on the coordinates x_1 and x_2 but does not depend explicitly on the coordinate x_3: $W = W(Y, L, x_1, x_2)$ (in this case, the material can be anisotropic). For the body homogeneous along the coordinate x_3, Eqs. (1.2) and (1.5) yield

$$
D(x_1, x_2, x_3) = D_0(x_1, x_2)\cdot Q(x_3), \qquad G(x_1, x_2, x_3) = G_0(x_1, x_2)\cdot Q(x_3). \tag{1.8}
$$

With allowance for (1.8), the equilibrium equations (1.1) for strains of the form (1.4), (1.5) become

$$
\nabla\cdot D_0 + \psi\boldsymbol{i}_3\cdot D_0\cdot e = 0; \tag{1.9}
$$

$$
\nabla\cdot G_0 + \psi\boldsymbol{i}_3\cdot G_0\cdot e + (C_0^{\mathrm{t}}\cdot D_0)_\times = 0. \tag{1.10}
$$

Here $e = -E\times\boldsymbol{i}_3$ is the discriminant tensor and ∇ is a plane gradient operator that is written in Cartesian coordinates as

$$
\nabla = \boldsymbol{i}_1\frac{\partial}{\partial x_1} + \boldsymbol{i}_2\frac{\partial}{\partial x_2}.
$$

Using the component representations of the tensors

$$
C_0 = C_{sk}\boldsymbol{i}_s\otimes\boldsymbol{i}_k, \qquad D_0 = D_{sk}\boldsymbol{i}_s\otimes\boldsymbol{i}_k, \qquad G_0 = G_{sk}\boldsymbol{i}_s\otimes\boldsymbol{i}_k,
$$

from Eqs. (1.9) and (1.10), we obtain the component form of the equilibrium equations for the torsion problem:

$$
\frac{\partial D_{11}}{\partial x_1} + \frac{\partial D_{21}}{\partial x_2} = \psi D_{32}, \qquad \frac{\partial D_{12}}{\partial x_1} + \frac{\partial D_{22}}{\partial x_2} = -\psi D_{31},
$$

$$\frac{\partial D_{13}}{\partial x_1} + \frac{\partial D_{23}}{\partial x_2} = 0; \tag{1.11}$$

$$\begin{gathered}\frac{\partial G_{11}}{\partial x_1} + \frac{\partial G_{21}}{\partial x_2} - \psi G_{32} + C_{12}D_{13} + C_{22}D_{23} + C_{32}D_{33} - C_{13}D_{12} - C_{23}D_{22} - C_{33}D_{32} = 0, \\ \frac{\partial G_{12}}{\partial x_1} + \frac{\partial G_{22}}{\partial x_2} + \psi G_{31} + C_{13}D_{11} + C_{23}D_{21} + C_{33}D_{31} - C_{11}D_{13} - C_{21}D_{23} - C_{31}D_{33} = 0, \\ \frac{\partial G_{13}}{\partial x_1} + \frac{\partial G_{23}}{\partial x_2} + C_{11}D_{12} + C_{21}D_{22} + C_{31}D_{32} - C_{12}D_{11} - C_{22}D_{21} - C_{32}D_{31} = 0.\end{gathered} \tag{1.12}$$

The proper orthogonal tensor H_0 can be expressed [4] in terms of the finite-rotation vector $\boldsymbol{\theta}$:

$$H_0 = \frac{1}{4+\theta^2}\,[(4-\theta^2)E + 2\boldsymbol{\theta}\otimes\boldsymbol{\theta} - 4E\times\boldsymbol{\theta}]. \tag{1.13}$$

Then, from Eqs. (1.2), (1.7), (1.8), and (1.13), it follows that Eqs. (1.11) and (1.12) constitute a system of six scalar equations for three functions of two variables $u_1(x_1, x_2)$, $u_2(x_1, x_2)$, and $w(x_1, x_2)$ and three components of the vector $\boldsymbol{\theta}$: $\theta_k(x_1, x_2) = \boldsymbol{\theta}\cdot\boldsymbol{i}_k$ $(k = 1, 2, 3)$. In the case where a distributed force load $\boldsymbol{f}$ and a distributed moment load $\boldsymbol{m}$ are applied to the lateral surface of the prism with the normal $\boldsymbol{n} = n_1\boldsymbol{i}_1 + n_2\boldsymbol{i}_2$, the boundary conditions on this surface are given by

$$\boldsymbol{n}\cdot D = \boldsymbol{f}, \qquad \boldsymbol{n}\cdot G = \boldsymbol{m}. \tag{1.14}$$

We assume that the external-load vectors are written as $\boldsymbol{f} = \boldsymbol{f}^*\cdot C$ and $\boldsymbol{m} = \boldsymbol{m}^*\cdot C$, where the vectors $\boldsymbol{f}^*$ and $\boldsymbol{m}^*$ do not depend on the coordinate x_3. (For example, the vector $\boldsymbol{f}^*$ does not depend on x_3 in the case of a hydrostatic pressure distributed uniformly over the lateral surface and the vector $\boldsymbol{m}^*$ does not depend on x_3 in the case of a uniform moment load directed normally to the deformed lateral surface of the cylindrical body.) Then, the boundary conditions (1.14) for strains of the form (1.4), (1.5) do not contain the variable x_3 and, together with Eqs. (1.9) and (1.10), constitute the two-dimensional boundary-value problem for a plane region shaped like the cross section of the prism.

Thus, the assumptions (1.4) and (1.5) on the nature of the deformation of the prismatic bar reduce the original three-dimensional nonlinear static problem for a Cosserat medium to a two-dimensional nonlinear boundary-value problem.

Let the lateral surface of the bar be stress free, i.e., $\boldsymbol{f} = \boldsymbol{m} = 0$. The boundary-value problem for a plane region σ shaped like the cross section of the bar comprises the equilibrium equations (1.9) and (1.10) and the boundary conditions on $\partial\sigma$

$$\boldsymbol{n}\cdot D_0 = 0, \qquad \boldsymbol{n}\cdot G_0 = 0 \tag{1.15}$$

[the tensors D_0 and G_0 in (1.9), (1.10), and (1.15) are expressed in terms of the unknown functions of two variables u_1, u_2, w, and $\boldsymbol{\theta}$ using the constitutive relations and formulas (1.3) and (1.7) and ψ and λ are specified constant parameters].

Let u_1, u_2, w, and H_0 be a certain solution of the boundary-value problem formulated above. One can show that the functions

$$\begin{gathered}u_1^* = u_1\cos\omega - u_2\sin\omega, \qquad u_2^* = u_1\sin\omega + u_2\cos\omega, \qquad w^* = w + d, \\ H_0^* = H_0\cdot(g\cos\omega + e\sin\omega + \boldsymbol{i}_3\otimes\boldsymbol{i}_3), \qquad g = E - \boldsymbol{i}_3\otimes\boldsymbol{i}_3\end{gathered} \tag{1.16}$$

(ω and d are arbitrary real constants) satisfy Eqs. (1.9) and (1.10) and boundary conditions (1.15). The insensitivity of the boundary-value problem of the cross section to the change of variables (1.16) implies that after deformation, the location of the elastic body is determined with accuracy up to the rotation about the X_3 axis and the translation along this axis. This ambiguity of the solution can be eliminated by imposing the following additional conditions on the unknown functions:

$$\iint\limits_{\sigma} w\,d\sigma = 0, \qquad \iint\limits_{\sigma} (\operatorname{tr} H_0 - 3)\,d\sigma = 0. \tag{1.17}$$

In this case, the problem (1.9), (1.10), (1.15) has a unique solution.

2. Forces and Moments Acting on the Ends of the Bar. The solution of the two-dimensional problem formulated above for the bar cross section satisfies the equilibrium equations inside the body and the boundary conditions on its lateral surface. The boundary conditions on the end surfaces of the cylinder $x_3 = \text{const}$ are satisfied only approximately in the Saint-Venant integral sense by choosing the constants ψ and λ.

We determine the principal vector $\boldsymbol{F}$ and the principal moment $\boldsymbol{M}$ of the forces and moments acting in an arbitrary cross section of a cylindrical body with a stress-free lateral surface subjected to torsional strain of the form (1.4), (1.5). Using (1.8), we obtain

$$\boldsymbol{F}(x_3) = \iint\limits_{\sigma} \boldsymbol{i}_3 \cdot D\, d\sigma = F_1\boldsymbol{e}_1 + F_2\boldsymbol{e}_2 + F_3\boldsymbol{i}_3,$$
$$\boldsymbol{M}(x_3) = \iint\limits_{\sigma} [\boldsymbol{i}_3 \cdot G - \boldsymbol{i}_3 \cdot D \times (u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2 + \lambda x_3\boldsymbol{i}_3 + w\boldsymbol{i}_3)]\, d\sigma = M_1\boldsymbol{e}_1 + M_2\boldsymbol{e}_2 + M_3\boldsymbol{i}_3, \tag{2.1}$$

where

$$F_k = \iint\limits_{\sigma} D_{3k}\, d\sigma, \qquad M_1 = \iint\limits_{\sigma} (G_{31} + D_{33}u_2 - D_{32}w)\, d\sigma,$$

$$M_2 = \iint\limits_{\sigma} (G_{32} + D_{31}w - D_{33}u_1)\, d\sigma, \qquad M_3 = \iint\limits_{\sigma} (G_{33} + D_{32}u_1 - D_{31}u_2)\, d\sigma.$$

[The principal moment in (2.1) is taken about the point $X_1 = X_2 = X_3 = 0$.] Considering the equilibrium of the region of the cylinder bounded by the lateral surface and the cross sections $x_3 = a$ and $x_3 = b$ (a and b are arbitrary real numbers) in the same way as in [5], we obtain

$$F_1 = F_2 = M_1 = M_2 = 0. \tag{2.2}$$

Equalities (2.2) imply that for strains of the form (1.4), (1.5) to occur, the ends of the cylinder should be loaded by a system of forces and moments that is statically equivalent to the force F_3 and the moment M_3 applied to a point on the X_3 axis and directed along this axis. It is also assumed that the cross section of the bar σ possesses central symmetry, i.e., it is brought into coincidence with itself by a rotation of 180° about the bar axis. An example is a Z-shaped cross section is. Doubly-symmetric cross sections also belong to this class. Using the method of [3] and assuming that the material is orthotropic, we can prove that the solutions of the two-dimensional boundary-value problem (1.9), (1.10), (1.15) possess the following property:

$$X_\alpha(-x_1, -x_2, x_3) = -X_\alpha(x_1, x_2, x_3) \qquad (\alpha = 1, 2). \tag{2.3}$$

From (2.3) it follows that a horizontal cross section of the deformed bar also possesses central symmetry such that the X_3 axis, i.e., the line $X_1 = X_2 = 0$ passes through the centers of all cross sections. In a particular case where the point $x_1 = x_2 = 0$ belongs to the region σ (i.e., the prismatic bar has no voids in the central part), relation (2.3) implies that $X_\alpha(0, 0, x_3) = 0$ ($\alpha = 1, 2$). This means that after torsion of the bar, the material straight line passing through the centers of cross sections of the undeformed bar remains a straight line and intersects the horizontal plane at the point $x_1 = x_2 = 0$.

Thus, for strains of the form (1.4), (1.5), the axial force F_3 arising at the ends of the bar of central-symmetric cross section passes through the cross-sectional center.

Once the two-dimensional boundary-value problem for the cross section is solved, the axial force and the torsional moment become known functions of the parameters ψ and λ:

$$F_3 = F(\psi, \lambda), \qquad M_3 = M(\psi, \lambda). \tag{2.4}$$

Inverting the functions F and M, one obtains the parameters ψ and λ for given values of the force F_3 and the moment M_3.

We consider the functional Π of the specific free energy (the energy per unit length) of the elastic bar that is calculated for the solution $u_\alpha(x_1, x_2, \psi, \lambda)$, $w(x_1, x_2, \psi, \lambda)$, $H_0(x_1, x_2, \psi, \lambda)$ of the two-dimensional problem (1.9), (1.10), (1.15), (1.17) and depends on the parameters ψ and λ:

$$\Pi(\psi, \lambda) = \iint_\sigma W[u_\alpha(x_1, x_2, \psi, \lambda), w(x_1, x_2, \psi, \lambda), H_0(x_1, x_2, \psi, \lambda); \psi, \lambda]\, d\sigma. \tag{2.5}$$

Relation (2.5) takes into account that according to (1.2), (1.6), and (1.7), the specific free energy depends on the parameters ψ and λ not only in terms of the functions u_α, w, and H_0 but also explicitly. Using the method described in a different context in [6], we can prove the following the energy relations of the nonlinear theory of torsion of cylindrical bodies with moment stresses:

$$F = \frac{\partial \Pi(\psi, \lambda)}{\partial \lambda}, \qquad M = \frac{\partial \Pi(\psi, \lambda)}{\partial \psi}. \tag{2.6}$$

Representations (2.6) describe the nonlinear interaction between axial and torsional strains in elastic cylinders of micropolar materials and allow one, in particular, to study direct and inverse Pointing effects [7] in these cylinders.

3. Compatibility Equations and Stress Functions. We transform the boundary-value problem (1.9), (1.10), (1.15), (1.17) for the prism cross section by eliminating the functions u_1, u_2, and w from it and using other quantities as the primary unknowns. As a result, for $C_{sk} = \boldsymbol{i}_s \cdot C_0 \cdot \boldsymbol{i}_k$, we obtain the following compatibility equations for the strain-gradient components:

$$\frac{\partial C_{32}}{\partial x_1} = \psi C_{11}, \qquad \frac{\partial C_{32}}{\partial x_2} = \psi C_{21}, \qquad \frac{\partial C_{31}}{\partial x_1} = -\psi C_{12}, \qquad \frac{\partial C_{31}}{\partial x_2} = -\psi C_{22}; \tag{3.1}$$

$$\frac{\partial C_{13}}{\partial x_2} = \frac{\partial C_{23}}{\partial x_1}. \tag{3.2}$$

With allowance for (1.7), Eqs. (3.1) and (3.2) are written in invariant coordinate-free form

$$\nabla \otimes \boldsymbol{i}_3 \cdot Y \cdot H_0 \cdot g = \psi g \cdot Y \cdot H_0 \cdot e; \tag{3.3}$$

$$\nabla \cdot e \cdot Y \cdot H_0 \cdot \boldsymbol{i}_3 = 0. \tag{3.4}$$

Using (3.3) and (3.4), we can write the compatibility equations in any curvilinear coordinates introduced in the region σ. Equations (3.3) and (3.4) supplemented by the equilibrium equations (1.9) and (1.10) and the relation $\boldsymbol{i}_3 \cdot Y \cdot H_0 \cdot \boldsymbol{i}_3 = \lambda$ constitute a complete system of equations with unknown functions Y and H_0. In this case, the first constraint (1.17) is not required to formulate the two-dimensional boundary-value problem in terms of Y and H_0.

As in the nonlinear theory of torsion ignoring moment stresses [5], the system of equations comprising the equilibrium and compatibility equations is appropriate for the case of a prismatic body with screw dislocations whose axes are parallel to the x_3 axis. The equilibrium conditions (1.11) for the force stresses can be satisfied identically by using the substitution

$$\begin{gathered} D_{\alpha\beta} = \psi \Phi_{\alpha\beta} \qquad (\alpha, \beta = 1, 2), \\ D_{13} = \frac{\partial \Omega}{\partial x_2}, \qquad D_{23} = -\frac{\partial \Omega}{\partial x_1}, \qquad D_{31} = -\frac{\partial \Phi_{12}}{\partial x_1} - \frac{\partial \Phi_{22}}{\partial x_2}, \qquad D_{32} = \frac{\partial \Phi_{11}}{\partial x_1} + \frac{\partial \Phi_{21}}{\partial x_2}, \end{gathered} \tag{3.5}$$

where $\Omega(x_1, x_2)$ and $\Phi_{\alpha\beta}(x_1, x_2)$ are stress functions. Expressions (3.5) are the general solution of Eqs. (1.11) since the functions $\Phi_{\alpha\beta}(x_1, x_2)$ $(\alpha, \beta = 1, 2)$ are uniquely determined for the stresses D_{sk} specified in the simply connected region σ, whereas the function Ω is determined with accuracy up to an additive constant that has no effect on the stresses in the body. For the simply connected region σ, the force boundary conditions (1.15) are written in terms of the stress functions:

$$\Omega = 0, \qquad n_\alpha \Phi_{\alpha\beta} = 0 \quad \text{on} \quad \partial\sigma \qquad (n_\alpha = \boldsymbol{n} \cdot \boldsymbol{i}_\alpha). \tag{3.6}$$

Introducing the stress-function tensor $\Phi = \Phi_{\alpha\beta} \boldsymbol{i}_\alpha \otimes \boldsymbol{i}_\beta$ $(\alpha, \beta = 1, 2)$, we write the general solution (3.5) of the force equilibrium equations in invariant form

$$D_0 = \psi \Phi + e \cdot \nabla \Omega \otimes \boldsymbol{i}_3 + \boldsymbol{i}_3 \otimes (\nabla \cdot \Phi) \cdot e + D_{33} \boldsymbol{i}_3 \otimes \boldsymbol{i}_3. \tag{3.7}$$

4. Variational Formulations of the Problem for the Cross Section. We consider the specific-energy functional defined on the set of functions $u_\alpha(x_1, x_2)$, $w(x_1, x_2)$, and $H_0(x_1, x_2)$ that are twice-differentiable in the region σ and satisfy conditions (1.17):

$$\Pi[u_1, u_2, w, H_0] = \iint\limits_\sigma W(Y, L)\, d\sigma. \tag{4.1}$$

According to (1.6) and (1.7), the specific free energy $W(Y, L)$ in (4.1) is expressed in terms of the functions u_1, u_2, w, and H_0. With allowance for (1.2) and (1.7), the variation of the functional (4.1) is given by

$$\delta\Pi = \iint\limits_\sigma \delta W\, d\sigma,$$

$$\delta W = \operatorname{tr}(D^{\mathrm t} \cdot \operatorname{grad} \delta\boldsymbol{R}) + \operatorname{tr}[D^{\mathrm t} \cdot (\operatorname{grad} \boldsymbol{R} \times \boldsymbol{\chi})] + \operatorname{tr}(G_0^{\mathrm t} \cdot \nabla\boldsymbol{\chi}) - \psi \boldsymbol{i}_3 \cdot G_0 \cdot e \cdot \boldsymbol{\chi},$$

$$\boldsymbol{R} = u_1 \boldsymbol{e}_1 + u_2 \boldsymbol{e}_2 + (\lambda x_3 + w)\boldsymbol{i}_3,$$

where $\boldsymbol{\chi}(x_1, x_2)$ is the virtual-rotation vector defined by the relation $H_0^{\mathrm t} \cdot \delta H_0 = -E \times \boldsymbol{\chi}$. One can verify that the equilibrium equations (1.9) and (1.10) written in terms of the functions u_1, u_2, w, and H_0 become Euler's equations of the variational problem of finding stationary values of the functional Π and the conditions on the lateral surface of the prismatic bar (1.15) become the natural boundary conditions of this problem.

As in the problem of nonlinear pure bending of bodies with moment stresses [8], one can obtain a variational formulation of the two-dimensional problem that is similar to the Castigliano principle in the classical theory of elasticity.

We consider the class of materials whose specific potential strain energy can be written as

$$W(Y, L) = W_1(Y) + W_2(L). \tag{4.2}$$

Condition (4.2) is satisfied, for example, for a physically linear isotropic Cosserat continuum [2] whose elastic potential is a quadratic form of the tensors $Y - E$ and L:

$$W = (1/2)[\lambda_0 \operatorname{tr}^2 \varepsilon + (\mu + \alpha) \operatorname{tr}(\varepsilon \cdot \varepsilon^{\mathrm t}) + (\mu - \alpha) \operatorname{tr} \varepsilon^2$$

$$+\beta \operatorname{tr}^2 L + (\gamma + \eta) \operatorname{tr}(L \cdot L^{\mathrm t}) + (\gamma - \eta) \operatorname{tr} L^2], \qquad \varepsilon = Y - E \tag{4.3}$$

(λ_0, μ, α, γ, β, and η are elastic constants). The absence of bilinear terms (i.e., terms that are linear in ε and in L) in (4.3) is due to the fact that the flexural-strain measure L is a pseudotensor and changes sign for space inversion. For materials possessing the property (4.2), Eq. (1.2) implies that the tensor P depends only on Y and the tensor K depends only on L:

$$P(Y) = \frac{dW_1(Y)}{dY}, \qquad K(L) = \frac{dW_2(L)}{dL}. \tag{4.4}$$

Assuming that the relation $P(Y)$ can be inverted uniquely, we construct a function $V_1(P)$ related to $W_1(Y)$ by the Legendre transformation:

$$V_1(P) = \operatorname{tr}[P^{\mathrm t} \cdot Y(P)] - W_1(P). \tag{4.5}$$

According to the property of the Legendre transformation, we have

$$Y = \frac{dV_1}{dP}. \tag{4.6}$$

The function

$$V(P, L) = V_1(P) + W_2(L) \tag{4.7}$$

will be called the specific complementary energy of elastic materials having the property (4.2). Relations (4.4)–(4.7) yield

$$Y = \frac{\partial V}{\partial P}, \qquad K = \frac{\partial V}{\partial L}. \tag{4.8}$$

As an example, we consider the following expression for the function of the specific complementary energy of a material with the potential (4.3):

$$V(P, L) = \operatorname{tr} P + \frac{1 + m}{8m\mu} \operatorname{tr}(P \cdot P^{\mathrm t}) - \frac{m - 1}{8m\mu} \operatorname{tr} P^2 - \frac{\nu}{4\mu(1 + \nu)} \operatorname{tr}^2 P + \frac{1}{2}[\beta \operatorname{tr}^2 L + (\gamma + \eta) \operatorname{tr}(L \cdot L^{\mathrm t}) + (\gamma - \eta) \operatorname{tr} L^2],$$

$$m = \frac{\lambda_0}{\mu}, \qquad \nu = \frac{\lambda_0}{2(\lambda_0 + \mu)}.$$

Because $P = D_0 \cdot H_0^{\mathrm{t}}$ for the torsion problem, according to (1.5), (1.7), and (1.8), and because the tensor L is expressed in terms of H_0 and ∇H_0, the specific complementary energy can be regarded as a function of D_0, H_0, and ∇H_0.

In the torsion problem of a prismatic body with moment stresses, the Castigliano-type functional is given by

$$\Pi_1(\Phi, \Omega, D_{33}, H_0) = \iint\limits_{\sigma} V[D_0(\Phi, \Omega, D_{33}), H_0, L(H_0)]\, d\sigma. \tag{4.9}$$

Expression (4.9) is based on representation (3.7) of the Piola stress tensor in terms of stress functions that identically satisfy the force equilibrium conditions (1.9). The admissible stress functions should be twice differentiable and satisfy boundary conditions (3.6), and the varied field of the orthogonal tensor H_0 should satisfy the second relation in (1.17).

With allowance for (4.8), the variation of functional (4.9) is written as

$$\delta\Pi_1 = \iint\limits_{\sigma} \Big\{ \mathrm{tr}\,[(Y \cdot H_0) \cdot \delta D_0^{\mathrm{t}}] + (C_0^{\mathrm{t}} \cdot D_0)_\times \cdot \boldsymbol{\chi} + \mathrm{tr}\,(G_0^{\mathrm{t}} \cdot \nabla \boldsymbol{\chi}) - \psi \boldsymbol{i}_3 \cdot G_0 \cdot e \cdot \boldsymbol{\chi} \Big\}\, d\sigma. \tag{4.10}$$

From (3.7) and (4.10) it follows that the stationarity condition $\delta\Pi_1 = 0$ is equivalent to the compatibility equations (3.3) and (3.4), the moment equilibrium equations (1.10), the relation $\lambda = \partial V / \partial D_{33}$, and the moment boundary conditions (1.15).

The variational formulations considered above can be used to solve the two-dimensional problem for the cross section of a prismatic body by the Ritz method or finite-element methods.

5. Torsion of a Circular Cylinder. We consider circular cylindrical coordinates: Lagrangian coordinates r, φ, and z and Eulerian coordinates R, Φ, and Z. The following formulas are valid:

$$x_1 = r\cos\varphi, \qquad x_2 = r\sin\varphi, \qquad x_3 = z,$$

$$X_1 = R\cos\Phi, \qquad X_2 = R\sin\Phi, \qquad X_3 = Z.$$

Using cylindrical coordinates, we write the strain family (1.4), (1.5) for a Cosserat continuum in equivalent form

$$R = \rho(r, \varphi), \qquad \Phi = \psi z + v(r, \varphi), \qquad Z = \lambda z + w(r, \varphi), \qquad H = H_0(r, \varphi) \cdot Q(z). \tag{5.1}$$

A particular case of representation (5.1) is the expression proposed in [9] for the torsional and axial tensile–compressive strains of a circular cylinder:

$$\begin{gathered} R = \rho(r), \qquad \Phi = \varphi + \psi z, \qquad Z = \lambda z, \\ H = \boldsymbol{e}_r \otimes \boldsymbol{e}_R + \cos\tau(r)(\boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\Phi + \boldsymbol{e}_z \otimes \boldsymbol{e}_Z) - \sin\tau(r)(\boldsymbol{e}_z \otimes \boldsymbol{e}_\Phi - \boldsymbol{e}_\varphi \otimes \boldsymbol{e}_Z), \\ \boldsymbol{e}_r = \boldsymbol{i}_1 \cos\varphi + \boldsymbol{i}_2 \sin\varphi, \qquad \boldsymbol{e}_\varphi = -\boldsymbol{i}_1 \sin\varphi + \boldsymbol{i}_2 \cos\varphi, \\ \boldsymbol{e}_R = \boldsymbol{i}_1 \cos\Phi + \boldsymbol{i}_2 \sin\Phi, \qquad \boldsymbol{e}_\Phi = -\boldsymbol{i}_1 \sin\Phi + \boldsymbol{i}_2 \cos\Phi, \\ \boldsymbol{e}_z = \boldsymbol{e}_Z = \boldsymbol{i}_3. \end{gathered} \tag{5.2}$$

As is proved in [9], the substitution (5.2) reduces the torsion problem for a circular (or hollow) cylinder made of an isotropic polar material to a boundary-value problem for a system of two nonlinear ordinary differential equations for the functions $\rho(r)$ and $\tau(r)$. For the case of an incompressible isotropic Cosserat pseudocontinuum, according to [9], the indicated problem admits an exact solution in quadratures.

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REFERENCES

1. L. I. Shkutin, *Deformation Mechanics of Flexible Bodies* [in Russian], Nauka, Novosibirsk (1988).
2. L. M. Zubov, *Nonlinear Theory of Dislocations and Disclinations in Elastic Bodies*, Springer, Berlin (1997).
3. L. M. Zubov and L. U. Bogachkova, "The theory of torsion of elastic noncircular cylinders under large deformations," *Trans. ASME, J. Appl. Mech.*, **62**, No. 2, 373–379 (1995).
4. L. M. Zubov, *Methods of the Nonlinear Theory of Elasticity in Shell Theory* [in Russian], Izd. Rostov. Gos. Univ., Rostov-on-Don (1982).
5. L. M. Zubov and A. V. Guba, "The nonlinear theory of torsion of prismatic elastic bodies containing screw dislocations," *Izv. Vyssh. Uchebn. Zaved., Sev.-Kavk. Reg., Estestv. Nauki*, Special Issue: *Nonlinear Problems of Solid Mechanics*, 212–222 (2003).
6. L. M. Zubov and A. A. Zelenina, "Three-dimensional analysis of large strains in a curved bar under spatial bending," *Ékol. Vestn. Nauch. Tsentr. Chernomor. Ekon. Sotr.*, No. 1, 46–54 (2003).
7. L. M. Zubov, "Direct and inverse Pointing effects in elastic cylinders," *Dokl. Ross. Akad. Nauk*, **380**, No. 2, 194–196 (2001).
8. A. A. Zelenina, "The nonlinear theory of pure bending of prismatic elastic bodies with moment stresses," *Izv. Vyssh. Uchebn. Zaved., Sev.-Kavk. Reg., Estestv. Nauki*, Special Issue: *Nonlinear Problems of Solid Mechanics*, 207–211 (2003).
9. A. V. Glushenko and L. M. Zubov, "Exact solutions of large-strain problems of nonlinear-elastic bodies with moment stresses," in: *Fundamental and Applied Problems of Mechanics of Deformable Solids and Structures* [in Russian], Vol. 1, Nizhnii Novgorod (1993), pp. 70–78.